Applications of Sadik Transform for solving Bessel's Function and linear Volterra Integral Equation of Convolution type

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***Abstract*: - In this article Sadik Transform of Bessel's function of order n of first kind is derived. This article demonstrated the applicability of Sadik Transform for solving Volterra Integral equation of convolution type with applications.**

***Keywords*: - Bessel's function, Sadik Transform, Inverse Sadik Transform, Volterra integral equation, convolution.**

1. Introduction

In daily life most of advance problems of basic Science and Engineering can represent mathematically in the form of Volterra integral equation. Integral Transform is very powerful Mathematical tool applied for solving differential equation and integral equation. Many transforms like, Laplace transform, Kamal Transform[2], Elazki Transform[3] etc. are used to solve Bessel's function[4] and Volterra integral equation. Sadik Latif Shaikh introduce new integral transform called as Sadik Transform [1].

1.1 Definition of Sadik Transform[1]:-

If $f(t)$ is piecewise continuous in $0 \le t \le A$ for any $A > 0$ and $|f(t)| \le Ke^{at}$ when $t \ge M$, for any real constant a and some positive constant K and M then Sadik Transform of $f(t)$ is defined by

$$F(v^\alpha, \beta) = S\{f(t)\} = \frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha} f(t)dt$$

Where v is complex variable, α is any non zero real number and β is any real number.

1.2 Linearity property of Sadik Transform:-

If $S\{f(t)\} = F(v^\alpha, \beta)$ and $S\{g(t)\} = G(v^\alpha, \beta)$ then

$S\{af(t) + bg(t)\} = aS\{f(t)\} + bS\{g(t)\} = aF(v^\alpha, \beta) + bG(v^\alpha, \beta)$ Where a and b are any constant.

Table I

Sadik Transform of some Elementary Functions:-

Sr. No.	f(t)	$S\{f(t)\} = F(v^\alpha, \beta)$
1	t^n	$\frac{n!}{v^{n\alpha+(\alpha+\beta)}}$, if n is positive integer
2	t^n	$\frac{\Gamma(n+1)}{v^{n\alpha+(\alpha+\beta)}}$
3	sinat	$\frac{av^{-\beta}}{v^{2\alpha}+a^2}$
4	cosat	$\frac{v^{\alpha-\beta}}{v^{2\alpha}+a^2}$
5	e^{at}	$\frac{v^{-\beta}}{v^{\alpha}-a}$
6	sinhat	$\frac{av^{-\beta}}{v^{2\alpha}-a^2}$
7	coshat	$\frac{v^{\alpha-\beta}}{v^{2\alpha}-a^2}$

1.3 Convolution Function:-

The convolution of the function $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \int_0^t f(t-u)g(u)du.$$

1.4 Convolution theorem for Sadik Transform:-

If $S\{f(t)\} = F(v^\alpha, \beta)$ and $S\{g(t)\} = G(v^\alpha, \beta)$ then

$$S\{f(t) * g(t)\} = S\left\{\int_0^t f(t-u)g(u)du\right\}$$

$$= v^\beta S\{f(t)\}S\{g(t)\} = v^\beta F(v^\alpha, \beta)G(v^\alpha, \beta)$$

Theorem:-If $S\{e^{at}\} = \frac{v^{-\beta}}{v^\alpha - a}$ then $S\{t^n e^{at}\} = \frac{n!v^{-n\beta}}{(v^\alpha - a)^{n+1}}$.

Proof:- For $n = 1$

$$S\{te^{at}\} = \frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha} te^{at}dt$$

$$= \frac{1}{v^\beta}\int_0^\infty e^{-(v^\alpha - a)t}tdt$$

$$= \frac{v^{-\beta}}{(v^\alpha - a)^2}$$

For $n = 2$

$$S\{t^2 e^{at}\} = \frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha} t^2 e^{at}dt$$

$$= \frac{1}{v^\beta}\int_0^\infty e^{-(v^\alpha - a)t}t^2 dt$$

$$= \frac{2v^{-\beta}}{(v^\alpha - a)}\int_0^\infty e^{-(v^\alpha - a)t}tdt$$

$$= \frac{2v^{-\beta}}{(v^\alpha - a)}S\{te^{at}\}$$

$$= \frac{2v^{-\beta}}{(v^\alpha - a)}\frac{v^{-\beta}}{(v^\alpha - a)^2}$$

$$= \frac{2!v^{-2\beta}}{(v^\alpha - a)^3}$$

Similarly for $n = 3$,

$$S\{t^3 e^{at}\} = \frac{3!v^{-3\beta}}{(v^\alpha - a)^4}$$

In general, $S\{t^n e^{at}\} = \frac{n!v^{-n\beta}}{(v^\alpha - a)^{n+1}}$

1.5 Inverse Sadik Transform:-

If $S\{f(t)\} = F(v^\alpha, \beta)$ then $f(t)$ is called inverse Sadik Transform of $F(v^\alpha, \beta)$ i.e. $S^{-1}\{F(v^\alpha, \beta)\} = f(t)$

Table II

Inverse Sadik Transform of some elementary functions:-

Sr. No.	$F(v^\alpha, \beta)$	$f(t) = S^{-1}F(v^\alpha, \beta)$
1	$\frac{1}{v^{n\alpha+(\alpha+\beta)}}$	$\frac{t^n}{n!}$
2	$\frac{v^{-\beta}}{v^{2\alpha} + a^2}$	$\frac{\sin at}{a}$
3	$\frac{v^{\alpha-\beta}}{v^{2\alpha} + a^2}$	$\cos at$
4	$\frac{v^{-\beta}}{v^\alpha - a}$	e^{at}
5	$\frac{v^{-\beta}}{v^{2\alpha} - a^2}$	$\frac{\sinh at}{a}$
6	$\frac{v^{\alpha-\beta}}{v^{2\alpha} - a^2}$	$\cosh at$
7	$\frac{v^{-n\beta}}{(v^\alpha - a)^{n+1}}$	$\frac{t^n e^{at}}{n!}$

2. *Bessel's function*

2.1 Definition: The Bessel's function of order n of first kind is defined as

$$J_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!}\left(\frac{t}{2}\right)^{n+2r} \tag{1}$$

For $n = 0,1,2$

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \cdot 4^2} - \frac{t^6}{2^2.4^2 \cdot 6^2} + \cdots \tag{2}$$

$$J_1(t) = \frac{t}{2} - \frac{t^3}{2^2 \cdot 4} + \frac{t^5}{2^2 \cdot 4^2} - \frac{t^6}{2^2 \cdot 4^2.6} + \cdots \tag{3}$$

$$J_2(t) = \frac{t^2}{2.4} - \frac{t^4}{2^2.4.6} + \frac{t^6}{2^2 4^2 \cdot 6.8} - \frac{t^8}{2^2.4^2.6^2.8.10} + \cdots \tag{4}$$

Relation between $J_0(t), J_1(t)$ and $J_2(t)$

$$J_0{}'(t) = -J_1(t) \tag{5}$$

$$J_2(t) = J_0(t) + 2J_0{}''(t) \tag{6}$$

2.2 Sadik Transform of Bessel's function of order n

$$S\{J_n(t)\} = \frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha} J_n(t)dt$$

$$= \frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha}\left\{\sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!}\left(\frac{t}{2}\right)^{n+2r}\right\}dt$$

$$= \frac{1}{v^\beta}\sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!2^{n+2r}}\int_0^\infty e^{-tv^\alpha} t^{n+2r}dt$$

$$= \frac{1}{v^\beta}\sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!2^{n+2r}} \frac{(n+2r)!}{v^{(n+2r+1)\alpha}} \tag{7}$$

2.3 Sadik Transform of Bessel's function of order zero

Using (7) $$S\{J_0(t)\} = \frac{1}{v^\beta}\sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2 2^{2r}} \frac{(2r)!}{v^{(2r+1)\alpha}} \tag{8}$$

$$= \frac{1}{v^{\alpha+\beta}}\sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2 2^{2r}} \frac{(2r)!}{v^{2r\alpha}}$$

$$= \frac{1}{v^{\alpha+\beta}}\left[1 + \frac{1}{v^{2\alpha}}\right]^{-1/2}$$

$$= \frac{v^{-\beta}}{\sqrt{1+v^{2\alpha}}} \tag{9}$$

2.4 Sadik Transform of Bessel's function of order one

Applying Sadik Transform on (5),

$$S\{J_0{}'(t)\} = -S\{J_1(t)\}$$

$$S\{J_1(t)\} = -\left[v^\alpha S\{J_0(t)\} - v^{-\beta} J_0(0)\right]$$

$$= -\left[v^\alpha \frac{v^{-\beta}}{\sqrt{1+v^{2\alpha}}} - v^{-\beta}\right], \qquad \because J_0(0) = 1$$

$$= v^{-\beta} + \frac{v^{\alpha-\beta}}{\sqrt{1+v^{2\alpha}}} \tag{10}$$

2.5 Sadik Transform of Bessel's function of order two

Applying Sadik Transform on (6),

$$S\{J_2(t)\} = S\{J_0(t) + 2J_0{}''(t)\} = S\{J_0(t)\} + 2S\{J''(t)\}$$

$$= \frac{v^{-\beta}}{\sqrt{1+v^{2\alpha}}} + 2\left\{v^{2\alpha}\frac{v^{-\beta}}{\sqrt{1+v^{2\alpha}}} - v^{-\beta}J'(0) - v^{\alpha-\beta}J_0(0)\right\}$$

$$= \frac{v^{-\beta}}{\sqrt{1+v^{2\alpha}}} + 2\left\{\frac{v^{2\alpha-\beta}}{\sqrt{1+v^{2\alpha}}} - v^{\alpha-\beta}\right\}, \qquad \because J_0(0) = 1$$

$$= \frac{1+2v^{2\alpha}-2v^\alpha\sqrt{1+v^{2\alpha}}}{v^\beta\sqrt{1+v^{2\alpha}}} \tag{11}$$

3. *Volterra Integral Equation:*

Volterra Integral equation of second kind:-

$$u(x) = f(x) + \lambda\int_a^x K(x,t)u(t)dt \tag{12}$$

Volterra Integral equation of first kind:-

$$f(x) = \lambda\int_a^x K(x,t)u(t)dt \tag{13}$$

Where a is fixed constant, $K(x,t), f(x)$ are known functions and $u(x)$ is unknown function, λ is non zero real and complex parameter.

3.1 Solution of Volterra Integral equation of second kind of convolution type:-

Consider the Volterra Integral equation

$$u(x) = f(x) + \lambda \int_0^x K(x-t)u(t)dt \quad (14)$$

Where $K(x-t)$ is difference kernel, since it is the difference of the function $x-t$

Using convolution,

$$u(x) = f(x) + \lambda\{k(x) * u(x)\}$$

Applying Sadik Transform on both sides

$$S\{u(x)\} = S\{f(x)\} + \lambda S\{k(x) * u(x)\}$$

$$= S\{f(x)\} + \lambda v^{\beta} S\{k(x)\}S\{u(x)\}, \text{ by theorem 2.4}$$

$$S\{u(x)\}\left[1 - \lambda v^{\beta} S\{k(x)\}\right] = S\{f(x)\}$$

$$S\{u(x)\} = \frac{S\{f(x)\}}{1-\lambda v^{\beta} S\{k(x)\}}$$

Applying inverse Sadik Transform

$$u(x) = S^{-1}\left\{\frac{S\{f(x)\}}{1-\lambda v^{\beta} S\{k(x)\}}\right\}$$

This is the required solution of equation (14)

3.2 Solution of Volterra Integral equation of first kind of convolution type:-

Consider the Volterra Integral equation

$$f(x) = \lambda \int_0^x K(x-t)u(t)dt \quad (15)$$

Where $K(x-t)$ is difference kernel, since it is the difference of the function $(x-t)$

Using convolution, $f(x) = \lambda\{k(x) * u(x)\}$

Applying Sadik Transform on both sides

$$S\{f(x)\} = \lambda S\{k(x) * u(x)\}$$

$$= \lambda v^{\beta} S\{k(x)\}S\{u(x)\}, \text{ by theorem 2.4}$$

$$S\{u(x)\} = \frac{S\{f(x)\}}{\lambda v^{\beta} S\{k(x)\}}$$

Applying inverse Sadik Transform

$$u(x) = S^{-1}\left\{\frac{S\{f(x)\}}{\lambda v^{\beta} S\{k(x)\}}\right\}$$

This is the required solution of equation (15).

3.3 Applications:

3.3.1 Consider the Volterra Integral equation of second kind

$$u(x) = x + 2\int_0^x \cos(x-t)u(t)dt \quad (16)$$

Using convolution, $u(x) = x + 2\cos x * u(x)$

Applying Sadik Transform on both sides

$$S\{u(x)\} = S\{x\} + S\{2\cos x * u(x)\}$$

$$= S\{x\} + v^{\beta} S\{2\cos x\}S\{u(x)\}, \text{ by theorem 2.4}$$

$$= \frac{1}{v^{\beta} v^{2\alpha}} + 2v^{\beta}\frac{v^{\alpha-\beta}}{v^{2\alpha}+1} S\{u(x)\}$$

$$S\{u(x)\}\left[1 - 2v^{\beta}\frac{v^{\alpha-\beta}}{v^{2\alpha}+1}\right] = \frac{1}{v^{\beta} v^{2\alpha}}$$

$$S\{u(x)\} = \frac{v^{2\alpha}+1}{v^{\beta} v^{2\alpha}(v^{2\alpha}-2v^{\alpha}+1)}$$

Applying inverse Sadik Transform

$$u(x) = S^{-1}\left\{\frac{v^{2\alpha}+1}{v^{\beta} v^{2\alpha}(v^{\alpha}-1)^2}\right\}$$

$$= S^{-1}\left\{\frac{1}{v^{\beta} v^{2\alpha}} + \frac{2}{v^{\beta} v^{\alpha}} - \frac{2}{v^{\beta}(v^{\alpha}-1)} + \frac{2}{v^{\beta}(v^{\alpha}-1)^2}\right\}, \text{ by partial fraction}$$

$$= x + 2 - 2e^x + 2xe^x, \text{ by theorem 2.5}$$

$$= x + 2 + 2(x-1)e^x$$

This is the required solution of equation (16).

3.3.2 Consider the Volterra Integral equation of second kind

$$u(x) = 3x^2 + \int_0^x \sin(x-t)u(t)dt \quad (17)$$

Using convolution, $u(x) = 3x^2 + \sin x * u(x)$

Applying Sadik Transform on both sides

$$S\{u(x)\} = S\{3x^2\} + S\{\sin x * u(x)\}$$

$$= 3\frac{2!}{v^{3\alpha+\beta}} + v^{\beta}S\{\sin x\}S\{u(x)\}, \text{ by theorem 2.4}$$

$$= \frac{6}{v^{3\alpha+\beta}} + v^{\beta}\frac{v^{-\beta}}{v^{2\alpha}+1}S\{u(x)\}$$

$$S\{u(x)\}\left[1 - v^{\beta}\frac{v^{-\beta}}{v^{2\alpha}+1}\right] = \frac{6}{v^{3\alpha+\beta}}$$

$$S\{u(x)\} = \frac{6(v^{2\alpha}+1)}{v^{3\alpha+\beta}v^{2\alpha}} = 6\left[\frac{1}{v^{3\alpha+\beta}} + \frac{1}{v^{5\alpha+\beta}}\right]$$

Applying inverse Sadik Transform

$$u(x) = S^{-1}\left\{6\left[\frac{1}{v^{3\alpha+\beta}} + \frac{1}{v^{5\alpha+\beta}}\right]\right\} = 6\left[\frac{x^2}{2!} + \frac{x^4}{4!}\right]$$

$$= 3x^2 + \frac{x^4}{4}.$$

This is the required solution of equation (17).

3.3.3 Consider the Volterra Integral equation of second kind

$$u(x) = 1 - 2x - 4x^2 + \int_0^x [3 + 6(x-t) - 4(x-t)^2]u(t)dt \quad (18)$$

Using convolution, $u(x) = 1 - 2x - 4x^2 + [3 + 6x - 4x^2] * u(x)$

Applying Sadik Transform on both sides

$$S\{u(x)\} = S\{1 - 2x - 4x^2\} + S\{[3 + 6x - 4x^2] * u(x)\}$$

$$= \frac{1}{v^{\alpha+\beta}} - 2\frac{1}{v^{2\alpha+\beta}} - 4\frac{2!}{v^{3\alpha+\beta}} + v^{\beta}S\{3 + 6x - 4x^2\}S\{u(x)\}, \text{ by theorem 2.4}$$

$$= \frac{1}{v^{\alpha+\beta}} - 2\frac{1}{v^{2\alpha+\beta}} - 4\frac{2!}{v^{3\alpha+\beta}} + v^{\beta}\left[\frac{3}{v^{\alpha+\beta}} + 6\frac{1}{v^{2\alpha+\beta}} - 4\frac{2!}{v^{3\alpha+\beta}}\right]S\{u(x)\}$$

$$S\{x\}\left[1 - v^{\beta}\left(\frac{3}{v^{\alpha+\beta}} + \frac{6}{v^{2\alpha+\beta}} - \frac{8}{v^{3\alpha+\beta}}\right)\right] = \frac{1}{v^{\alpha+\beta}} - \frac{2}{v^{2\alpha+\beta}} - \frac{8}{v^{3\alpha+\beta}}$$

$$S\{x\} = \frac{v^{2\alpha} - 2v^{\alpha} - 8}{v^{\beta}(v^{3\alpha} - 3v^{2\alpha} - 6v^{\alpha} + 8)}$$

$$= \frac{1}{v^{\beta}}\frac{(v^{\alpha}-4)(v^{\alpha}+2)}{(v^{\alpha}-1)(v^{\alpha}+2)(v^{\alpha}-4)}$$

Applying inverse Sadik Transform

$$u(x) = S^{-1}\left\{\frac{v^{-\beta}}{v^{\alpha}-1}\right\} = e^{-x}.$$

This is the required solution of equation (18).

3.3.4 Consider the Volterra Integral equation of first kind

$$\sqrt{x} = \int_0^x \frac{\emptyset(t)}{\sqrt{x-t}}dt, \quad (19)$$

Using convolution, $\sqrt{x} = x^{-1/2} * \emptyset(x)$

Applying Sadik Transform on both sides

$$S\left\{x^{\frac{1}{2}}\right\} = S\left\{x^{-\frac{1}{2}} * \emptyset(x)\right\} = v^{\beta}S\left\{x^{-\frac{1}{2}}\right\}S\{\emptyset(x)\}, \text{ by theorem 2.4}$$

$$\frac{\Gamma\left(\frac{1}{2}+1\right)}{v^{\frac{3\alpha}{2}+\beta}} = v^{\beta}\frac{\Gamma\left(-\frac{1}{2}+1\right)}{v^{\frac{\alpha}{2}+\beta}}S\{\emptyset(x)\}$$

$$\frac{\frac{1}{2}\sqrt{\pi}}{v^{\frac{3\alpha}{2}+\beta}} = v^{\beta}\frac{\sqrt{\pi}}{v^{\frac{\alpha}{2}+\beta}}S\{\emptyset(x)\}$$

$$S\{\emptyset(x)\} = \frac{v^{\frac{\alpha}{2}+\beta}}{2v^{\beta}v^{\frac{3\alpha}{2}+\beta}} = \frac{1}{2}\frac{1}{v^{\alpha-\beta}}$$

Applying inverse Sadik Transform

$$\emptyset(x) = S^{-1}\left\{\frac{1}{2}\frac{1}{v^{\alpha-\beta}}\right\} = \frac{1}{2}$$

This is the required solution of equation (19).

3.3.5 Consider the Volterra Integral equation of first kind

$$\sin x = \int_0^x e^{x-t}u(t)dt, \quad (20)$$

Using convolution, $= e^x * u(x)$

Applying Sadik Transform on both sides

$$S\{\sin x\} = S\{e^x * u(x)\} = v^{\beta}S\{e^x\}S\{u(x)\}, \text{ by theorem 2.4}$$

$\frac{v^{-\beta}}{v^{2\alpha}+1} = v^{\beta}\frac{v^{-\beta}}{v^{\alpha}-1}S\{u(x)\}$

$S\{u(x)\} = \frac{v^{-\beta}(v^{\alpha}-1)}{v^{2\alpha}+1}$

Applying inverse Sadik Transform

$$u(x) = S^{-1}\left\{\frac{v^{\alpha-\beta}}{v^{2\alpha}+1} - \frac{v^{-\beta}}{v^{2\alpha}+1}\right\} = \cos x - \sin x$$

This is the required solution of equation (20).

3.3.6 Consider the Volterra Integral equation of first kind

$$x^2 = \frac{1}{2}\int_0^x (x-t)u(t)dt \quad (21)$$

Using convolution, $x^2 = \frac{1}{2}[x * u(x)]$

Applying Sadik Transform on both sides

$S\{x^2\} = \frac{1}{2}S\{[x * u(x)]\} = \frac{1}{2}v^{\beta}S\{x\}S\{u(x)\}$, by theorem 2.4

$\frac{2!}{v^{3\alpha+\beta}} = \frac{1}{2}v^{\beta}\frac{1}{v^{2\alpha+1}}S\{u(x)\}$

$S\{u(x)\} = 4\frac{1}{v^{\alpha+\beta}}$

Applying inverse Sadik Transform

$u(x) = 4S^{-1}\left\{\frac{1}{v^{\alpha+\beta}}\right\} = 4,$

This is the required solution of equation (21).

3.3.7 Consider the Volterra Integral equation of first kind

$$\sin x = \int_0^x u(x-t)u(t)dt, \quad (22)$$

Using convolution, $= u(x) * u(x)$

Applying Sadik Transform on both sides

$S\{\sin x\} = S\{u(x) * u(x)\} = v^{\beta}S\{u(x)\}S\{u(x)\}$, by theorem 2.4

$\frac{v^{-\beta}}{v^{2\alpha}+1} = v^{\beta}[S\{u(x)\}]^2$

$S\{u(x)\} = \pm\frac{v^{-\beta}}{\sqrt{v^{2\alpha}+1}}$

Applying inverse Sadik Transform

$$u(x) = \pm S^{-1}\left\{\frac{v^{-\beta}}{\sqrt{v^{2\alpha}+1}}\right\} = \pm J_0(x)$$

This is the required solution of equation (22).

3.3.8 Consider the Volterra Integral equation of first kind

$$1 - J_0(x) = \int_0^x u(t)dt, \quad (23)$$

Using convolution, $= 1 * u(x)$

Applying Sadik Transform on both sides

$S\{1 - J_0(x)\} = S\{1 * u(x)\} = v^{\beta}S\{1\}S\{u(x)\}$, by theorem 2.4

$\frac{1}{v^{\alpha+\beta}} - \frac{v^{-\beta}}{\sqrt{v^{2\alpha}+1}} = v^{\beta}\frac{1}{v^{\alpha+\beta}}S\{u(x)\}$

$S\{u(x)\} = v^{-\beta} - \frac{v^{\alpha-\beta}}{\sqrt{v^{2\alpha}+1}}$

Applying inverse Sadik Transform

$$u(x) = S^{-1}\left\{v^{-\beta} - \frac{v^{\alpha-\beta}}{\sqrt{v^{2\alpha}+1}}\right\} = J_1(x).$$

This is the required solution of equation (23).

3.3.9 Consider the Volterra Integral equation of first kind

$$x = \int_0^x e^{x-t}u(t)dt, \quad (24)$$

Using convolution, $= e^x * u(x)$

Applying Sadik Transform on both sides

$$S\{x\} = S\{e^x * u(x)\} = v^{\beta} S\{e^x\} S\{u(x)\}, \quad \text{by theorem 2.4}$$

$$\frac{1}{v^{2\alpha+\beta}} = v^{\beta} \frac{v^{-\beta}}{v^{\alpha-1}} S\{u(x)\}$$

$$S\{u(x)\} = \frac{v^{\alpha} - 1}{v^{2\alpha+\beta}}$$

Applying inverse Sadik Transform

$$u(x) = S^{-1}\left\{\frac{1}{v^{\alpha} - \beta} - \frac{1}{v^{2\alpha+\beta}}\right\} = 1 - x$$

This is the required solution of equation (24).

4. Conclusion:-

The applications of new integral transform "Sadik Transform" to the solution of Bessel's function and Volterra integral equation of convolution type has been demonstrated.

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