Solution of Nonhomogeneous Wave Equation Using Green's Function

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Abstract: **Application of Green's function has been very interesting among all popular methods to solve partial differential equations defined on infinite or semi-infinite domains. Each method to solve partial differential equations, has its own beauty and own limitations. Here in this paper, the intention is to show how elegantly a non – homogeneous wave equation defined on semi-infinite domain, can be solved using Green's function Method. This method deals with non-homogeneous boundary conditions directly, without converting those into homogeneous ones. All, what is required is to manage that the Green's function, obtained, satisfies corresponding homogenous boundary condition. With this purpose, firstly theory how to get Green's function for a wave equation, in n-dimensional infinite space is discussed [1], then using method of images, how to modify solution for semi-infinite domain [1], is discussed and using this derivation, one such one-dimensional problem is solved.**

Keywords: **Dirac delta function, Unit step function, wave operator, Green's formula, Method of images, Fourier Transform and inverse Fourier Transform.**

I. INTRODUCTION

Consider the $n-$dimensional, infinite domain, inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u + Q(X,t) \tag{1}$$

With initial conditions, $u(X,0) = f(X)$ (1-a)

$$\frac{\partial u}{\partial t}(X,0) = g(X) \qquad \text{(1-b)}$$

where $X = (x_1, x_2, x_3, \dots . x_n)$ and $\nabla^2 = \frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \cdots + \frac{\partial^2}{\partial {x_n}^2}$

To find the appropriate Green's function, for this infinite domain problem, it is assumed that the required Green's function $G(X,t;\, X_0, t_0)$, that is, the response at point X, at time t, due to a source located at point $X_0 = (x_{01}, x_{02}, x_{03}, \dots . x_{0n})$ at time t_0, is a solution of

$$\frac{\partial^2 G}{\partial t^2} = c^2\nabla^2 G + \delta(X - X_0)\delta(t - t_0), \tag{2}$$

where $\delta(X - X_0)$ is the n- dimensional Dirac delta function.

One- dimensional Dirac delta function [2], is defined, as $\delta(x) = \begin{cases} 0 & if\ x \neq 0 \\ \infty & if\ x = 0 \end{cases}$, (3)

with the properties,

(i) $\int_{-\infty}^{\infty} \delta(x)dx = 1$,

(ii) $\int_{-\infty}^{\infty} \varphi(x)\delta(x)dx = \varphi(0)$,

(iii) $\int_{-\infty}^{\infty} \delta(ax - b)\varphi(x)dx = |a|^{-1}\, \varphi(ba^{-1})$, where φ is any continuous function and (iv) $\delta(x) = \delta(-x)$.

Also, δ function is the derivative of the Heaviside unit step function $H(x)$,[2] where $H(x)$ is defined as

$$H(x) = \begin{cases} 0 & if\ x < 0 \\ 1 & if\ x > 0 \end{cases} \tag{4}$$

As the Green's function is the response at position X at time t, due to a source located at X_0 at time t_0, $G(X,t;X_0,t_0) = 0$ for $t < t_0$ (This result is known as Causality Principle)

Let L be the linear differential operator defined as, $L = \frac{\partial^2}{\partial t^2} - c^2\,\nabla^2$ (5)

So (1) converts into $L(u) = Q(X,t)$ and (2) converts into $L(G(X,t;X_0,t_0)) = \delta(X - X_0)\delta(t - t_0)$.

Also, $uL(v) - vL(u) = u\frac{\partial^2 v}{\partial t^2} - v\frac{\partial^2 u}{\partial t^2} - c^2\,(u\nabla^2 v - v\nabla^2 u)$ (6)

The Green's formula[1] for three dimensional Laplacian operator , that is, for $L = \nabla^2$, is

$\iiint [uL(v) - vL(u)]\, d^3x = \oiint (u\,\nabla v - v\,\nabla u)\cdot\hat{n}\, dS,$ (7)

For this wave operator (5), considering dimension $n = 3$ and using (6) and (7), one gets:

$$\int_{t_i}^{t_f}\left(\iiint [uL(v) - vL(u)]\, d^3x\right) dt = \int_{t_i}^{t_f}\left(\iiint \left(u\frac{\partial^2 v}{\partial t^2} - v\frac{\partial^2 u}{\partial t^2} - c^2\,(u\nabla^2 v - v\nabla^2 u)\right) d^3x\right) dt$$

$$= \iiint \int_{t_i}^{t_f}\left(u\frac{\partial^2 v}{\partial t^2} - v\frac{\partial^2 u}{\partial t^2}\right) dt\, d^3x - c^2\int_{t_i}^{t_f}\left(\oiint (u\,\nabla v - v\,\nabla u)\cdot\hat{n}\, dS\right) dt$$

$$= \iiint \left(\left[u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right]_{t_i}^{t_f} d^3x\right) - c^2\int_{t_i}^{t_f}\left(\oiint (u\,\nabla v - v\,\nabla u)\cdot\hat{n}\, dS\right) dt \quad (8)$$

where t_i denotes initial time, t_f denotes final time and $d^3x = dxdydz$.

Now, in (8), let $u = u(X,t)$, where $u(X,t)$ is solution of problem stated in (1) and $v = G(X_0,t_0;X,t)$.

As $G(X,t;X_0,t_0)$ expresses the response at X, at time t due to a source at X_0, acting instantaneously only at $t = t_0$, it will be the same as the response at X_0 at time t_0 due to a source at X, acting instantaneously only at time t, as far as the elapsed times from the sources are the same. That is, $G(X,t;X_0,t_0) = G(X_0,t_0;X,t)$. This property is known as reciprocity.

So, here $v = G(X,t_0;X_0,t) = G(X_0,t_0;X,t)$ (9)

As $u(X,t)$ is solution of problem stated in (1), along with initial conditions (1-a) and (1-b),

$$L(u) = Q(X,t) \quad \text{and} \quad L[G(X,t_0;X_0,t)] = \delta(X - X_0)\delta(t - t_0).$$

Hence, taking $t_i = 0$ and $t_f = t_{0^+}$, (7) implies that,

$$\int_0^{t_{0^+}}\left(\iiint \left[u(X,t)L\big(G(X,t_0;X_0,t)\big) - G(X,t_0;X_0,t)L(u)\right] d^3x\right) dt$$

$$= \iiint \left(\left[u\frac{\partial G}{\partial t} - G\frac{\partial u}{\partial t}\right]_0^{t_{0^+}} d^3x\right) - c^2\int_0^{t_{0^+}}\left(\oiint (u\,\nabla G - G\,\nabla u)\cdot\hat{n}\, dS\right) dt$$

$$\Rightarrow \int_0^{t_{0^+}}\left(\iiint [u(X,t)\,\delta(X - X_0)\delta(t - t_0) - G(X,t_0;X_0,t)Q(X,t)]\, d^3x\right) dt$$

$$= \iiint \left[\frac{\partial u}{\partial t}(X,0)G(X,t_0;X_0,0) - u(X,0)\frac{\partial}{\partial t}G(X,t_0;X_0,0)\right] d^3x$$

$$- c^2\int_0^{t_{0^+}}\left[\oiint \big(u(X,t)\nabla G(X_0,t_0;X,t) - G(X_0,t_0;X,t)\nabla u(X,t)\big)\cdot\hat{n}\, dS\right] dt$$

as $G(X,t_0;X_0,t_{0^+}) = 0$ and $\frac{\partial}{\partial t}G(X,t_0;X_0,t_{0^+}) = 0$.

Using property of Dirac delta function and (9), the above result simplifies to,

$$u(X_0,t_0) = \int_0^{t_{0^+}}\left(\iiint G(X_0,t_0;X,t)Q(X,t)d^3x\right)dt$$

$$+ \iiint \left[\frac{\partial u}{\partial t}(X,0)G(X_0,t_0;X,0) - u(X,0)\frac{\partial}{\partial t}G(X_0,t_0;X,0)\right] d^3x$$

$$- c^2\int_0^{t_{0^+}}\left[\oiint \big(u(X,t)\nabla G(X_0,t_0;X,t) - G(X_0,t_0;X,t)\nabla u(X,t)\big)\cdot\hat{n}\, dS\right] dt$$

Interchanging X with X_0, and t with t_0, the solution $u(X,t)$, in terms of $G(X,t;X_0,t_0)$ is:

$$u(X,t) = \int_0^{t_{0^+}}\left(\iiint G(X,t;X_0,t_0)Q(X_0,t_0)d^3x_0\right)dt_0$$

$$+ \iiint \left[\frac{\partial u}{\partial t_0}(X_0,0)G(X,t;X_0,0) - u(X_0,0)\frac{\partial}{\partial t_0}G(X,t_0;X_0,0)\right] d^3x_0$$

$$- c^2\int_0^{t_{0^+}}\left[\oiint \big(u(X_0,t_0)\nabla G(X,t;X_0,t_0) - G(X,t;X_0,t_0)\nabla u(X_0,t_0)\big)\cdot\hat{n}\, dS\right] dt_0 \quad (10)$$

Thus the relationship between the solution $u(X,t)$ and the Green's function $G(X,t;X_0,t_0)$ is established, now it remains only to obtain $G(X,t;X_0,t_0)$.

To determine$G(X,t;X_0,t_0)$, on taking Fourier transform of (2) one gets:

$$\mathcal{F}\left(\frac{\partial^2 G}{\partial t^2} - c^2\nabla^2 G\right) = \mathcal{F}\big(\delta(X - X_0)\delta(t - t_0)\big) \quad (11)$$

where Fourier Transform [1] (one dimensional) $\mathcal{F}(\omega)$ of a piece-wise continuous function $f(x)$, is defined as

$$\mathcal{F}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx$$

and inverse Fourier Transform $\mathcal{F}^{-1}$ of $\mathcal{F}(\omega)$, is defined as ;

$$\mathcal{F}^{-1}\big(\mathcal{F}(\omega)\big) = f(x) = \int_{-\infty}^{\infty}\mathcal{F}(\omega)e^{-i\omega x}\, d\omega$$

For three dimensional space, $\mathcal{F}(\boldsymbol{\omega}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} f(X)e^{i\omega X}d^3x$ and

$\mathcal{F}^{-1}(\mathcal{F}(\boldsymbol{\omega})) = f(X) = \int_{-\infty}^{\infty} \mathcal{F}(\boldsymbol{\omega}) e^{-i\omega X}\, d^3\omega$, where $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ and $X = (x_1, x_2, x_3)$

The L.H.S. of (11), converts into,

$\mathcal{F}\left(\frac{\partial^2 G}{\partial t^2} - c^2\nabla^2 G\right) = \frac{\partial^2}{\partial t^2}(\mathcal{F}(G) - c^2\nabla^2(\mathcal{F}(G)) = \frac{\partial^2}{\partial t^2}\bar{G} - c^2 i^2\omega^2\bar{G} = \frac{\partial^2}{\partial t^2}\bar{G} + c^2\omega^2\bar{G},$

where $\mathcal{F}(G) = \bar{G}(\boldsymbol{\omega}, t; X_0, t_0)$ and $\boldsymbol{\omega}\cdot\boldsymbol{\omega} = \omega^2$

The R.H.S of (11) converts into

$$\mathcal{F}(\delta(X - X_0)\delta(t - t_0)) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \delta(X - X_0)\delta(t - t_0)\, e^{i\omega X} d^3x = \frac{e^{i\omega X_0}}{(2\pi)^3}\,\delta(t - t_0),$$

(using property of δ)

Thus equation (11) converts into equation:

$$\frac{\partial^2}{\partial t^2}\bar{G}(\boldsymbol{\omega}, t; X_0, t_0) + c^2\omega^2\bar{G}(\boldsymbol{\omega}, t; X_0, t_0) = \frac{e^{i\omega X_0}}{(2\pi)^3}\,\delta(t - t_0) \tag{12}$$

$\bar{G}(\boldsymbol{\omega}, t; X_0, t_0)$ also satisfies causality principle, that is, $\bar{G}(\boldsymbol{\omega}, t; X_0, t_0) = 0$ for $t < t_0$.

As $\delta(t - t_0) = 0$ for $t > t_0$, for $t > t_0$, (12) converts into $\frac{\partial^2\bar{G}}{\partial t^2} + c^2\omega^2\bar{G} = 0$ (13)

Solving (13), one gets $\bar{G} = \begin{cases} 0 & t < t_0 \\ A\cos c\omega(t - t_0) + B\sin c\omega(t - t_0) & t > t_0 \end{cases}$

Since $\bar{G}(\boldsymbol{\omega}, t; X_0, t_0) = 0$ for $t < t_0$ and $\bar{G}$ is continuous at $t = t_0$, $\bar{G}(\boldsymbol{\omega}, t_0; X_0, t_0) = 0 \Rightarrow A = 0$

This implies $\bar{G} = \begin{cases} 0 & t < t_0 \\ B\sin c\omega(t - t_0) & t > t_0 \end{cases}$ (14)

To determine$B$, by integrating (12), from t_{0^-} to t_{0^+}, one gets,

$$\left[\frac{\partial\bar{G}}{\partial t}\right]_{t_{0^-}}^{t_{0^+}} + c^2\omega^2\int_{t_{0^-}}^{t_{0^+}}\bar{G}\, dt = \int_{t_{0^-}}^{t_{0^+}}\frac{e^{i\omega X_0}}{(2\pi)^3}\,\delta(t - t_0)\, dt = \int_{t_{0^-}}^{t_{0^+}}\frac{e^{i\omega X_0}}{(2\pi)^3}\left(\frac{d}{dt}(H(t - t_0))\right) dt$$

$$\Rightarrow \left[\frac{\partial\bar{G}}{\partial t}\right]_{t_{0^-}}^{t_{0^+}} + c^2\omega^2\int_{t_{0^-}}^{t_{0^+}}\bar{G}\, dt = \frac{e^{i\omega X_0}}{(2\pi)^3}[H(t - t_0)]_{t_{0^-}}^{t_{0^+}} = \frac{e^{i\omega X_0}}{(2\pi)^3}$$

$$\Rightarrow \frac{\partial\bar{G}}{\partial t}(\boldsymbol{\omega}, t_{0^+}; X_0, t_0) = \frac{e^{i\omega X_0}}{(2\pi)^3} \Rightarrow Bc\omega = \frac{e^{i\omega X_0}}{(2\pi)^3} \quad \text{as} \quad \frac{\partial\bar{G}}{\partial t}(\boldsymbol{\omega}, t; X_0, t_0) = c\omega B\cos(t - t_0)$$

$$\Rightarrow B = \frac{e^{i\omega X_0}}{c\omega(2\pi)^3}$$

So, (14) implies, $\bar{G}(\boldsymbol{\omega}, t; X_0, t_0) = \frac{e^{i\omega X_0}}{c\omega(2\pi)^3}\sin c\omega(t - t_0)$ for $t > t_0$.

$$\Rightarrow G(X, t; X_0, t_0) = \int_{-\infty}^{\infty}\bar{G}(\boldsymbol{\omega}, t; X_0, t_0)e^{-i\omega X}\, d^3\omega = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{e^{-i\omega(X - X_0)}\sin c\omega(t - t_0)}{c\omega}\, d^3\omega \tag{15}$$

Thus (15) provides required Green's function $G(X, t; X_0, t_0)$ for $t > t_0$. Substituting it, in result (10), one can obtain solution for the wave equation (1), with $n = 3$.

II. IMPLIMENTATION

Eq. (15) provides the Green's function $G(X, t; X_0, t_0)$ for three dimensional wave equation for infinite domain. Using this infinite space Green's function, one can easily obtain Green's function for semi-infinite domain [1] also. For that, the method of images is utilized. To illustrate the method, the following problem will be discussed and solved.

Problem:1 $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$; $x > 0$ (16)

with $u(x, 0) = 0$ (16-a)

$\frac{\partial u}{\partial t}(x, 0) = 0$ (16-b)

and $u(0, t) = h(t)$ (16-c)

The problem is regarding to find solution of one dimensional wave equation (16) for semi-infinite domain, as it involves only positive x. For that, solution for infinite domain should be modified accordingly.

If domain were given as $-\infty < x < \infty$, that is an infinite domain, then (15) provides $G(x, t; x_0, t_0)$, as:

$$G(x, t; x_0, t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega(x - x_0)}\sin c\omega(t - t_0)}{c\omega}\, d\omega, \text{ (here } n = 1) \tag{17}$$

As $\int_{-\infty}^{\infty}\frac{e^{-i\omega x}\sin c\omega(t - t_0)}{\pi\omega}\, d\omega = \begin{cases} 0 & |x| > c(t - t_0) \\ 1 & |x| < c(t - t_0) \end{cases}$

And as $\mathcal{F}^{-1}\left(e^{i\omega x_0}\mathcal{F}(\omega)\right) = f(x - x_0)$, if $\mathcal{F}$ is Fourier transform of f, (17) simplifies to,

$$G(x,t;x_0,t_0) = \frac{1}{2c}\int_{-\infty}^{\infty}\frac{e^{-i\omega(x-x_0)}\sin c\omega(t-t_0)}{\pi\omega}\,d\omega = \begin{cases}0 & |x-x_0| > c(t-t_0)\\ \frac{1}{2c} & |x-x_0| > c(t-t_0)\end{cases}$$

Expressing $G(x,t;x_0,t_0)$ in terms of Heaviside unit step function,

$$G(x,t;x_0,t_0) = \frac{1}{2c}\{-H[(x-x_0) - c(t-t_0)] + H[(x-x_0) + c(t-t_0)]\} \quad (18)$$

Now to obtain Green's function for semi-infinite domain, and as semi-infinite domains possess a boundary, here, it is $x = 0$, it is imagined that there is an infinite space problem with source $\delta(x-x_0)\delta(t-t_0)$ at $x = x_0$ and a negative image source $-\delta(x+x_0)\delta(t-t_0)$ at $x = -x_0$, to make value of Green's Function equal to zero at the boundary $x = 0$.

Hence, now taking $L(G(x,t;x_0,t_0)) = \delta(x-x_0)\delta(t-t_0) - \delta(x+x_0)\delta(t-t_0)$, and using (18),

$$G(x,t;x_0,t_0) = \frac{1}{2c}\{-H[(x-x_0) - c(t-t_0)] + H[(x-x_0) + c(t-t_0)]\}$$
$$-\frac{1}{2c}\{-H[(x+x_0) - c(t-t_0)] + H[(x+x_0) + c(t-t_0)]\} \quad (19)$$

One can easily verify, that $G(x,t;\,0,t_0) = 0$.

That is G satisfies homogeneous boundary condition at boundary, as required.

According to (10),

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty}(G(x,t;x_0,t_0)\cdot Q(x,t))\,dx_0\,dt_0$$
$$+\int_{-\infty}^{\infty}\left[\frac{\partial u}{\partial t_0}(x_0,0)G(x,t\,;x_0,0) - u(x_0,0)\frac{\partial}{\partial t_0}G(x,t_0;x_0,0)\right]dx_0$$
$$-c^2\int_0^t[\,u(0,t_0)\left[\frac{\partial}{\partial x_0}G(x,t;x_0,t_0)\right]_{x_0=0} - G(x,t;\,0,t_0)\left[\frac{\partial}{\partial x_0}u(x_0,t_0)\right]_{x_0=0}]\cdot\hat{n}\,dt_0$$

As given in (16), for this semi-infinite problem,

$Q(x,t) = 0,\ u(x,0) = f(x) = 0,\ \frac{\partial u}{\partial t}(x,0) = g(x) = 0,\ u(0,t) = h(t)$ and here $\hat{n} = -\hat{\imath}$.

So, $$u(x,t) = c^2\int_0^t h(t_0)\left[\frac{\partial}{\partial x_0}G(x,t;x_0,t_0)\right]_{x_0=0}dt_0 \quad (20)$$

Now $$\frac{\partial}{\partial x_0}G(x,t;x_0,t_0) = \frac{\partial}{\partial x_0}\left\{\frac{1}{2c}\{-H[(x-x_0) - c(t-t_0)] + H[(x-x_0) + c(t-t_0)]\}\right\}$$

$$-\frac{\partial}{\partial x_0}\left\{\frac{1}{2c}\{-H[(x+x_0) - c(t-t_0)] + H[(x+x_0) + c(t-t_0)]\}\right\}$$

$$\Rightarrow \frac{\partial}{\partial x_0}G(x,t;x_0,t_0) = \frac{1}{2c}\{\delta[(x-x_0) - c(t-t_0)] - \delta[(x-x_0) + c(t-t_0)]\}$$
$$-\frac{1}{2c}\{-\delta[(x+x_0) - c(t-t_0)] + \delta[(x+x_0) + c(t-t_0)]\} \text{ as } H'(x) = \delta(x).$$

$$\Rightarrow \left[\frac{\partial}{\partial x_0}G(x,t;x_0,t_0)\right]_{x_0=0} = \frac{1}{c}\{\delta[x - c(t-t_0)] - \delta[x + c(t-t_0)]\} = \frac{1}{c}\{\delta[x - c(t-t_0)]\}$$

as $c > 0,\ t > 0, x > 0 \Rightarrow x + ct > 0$ and as $0 < t_0 < t,\ \delta[x + c(t-t_0)] = 0$

Substituting this result in (20),

$$u(x,t) = c^2\int_0^t[h(t_0)\frac{1}{c}\{\delta[x - c(t-t_0)]\}\,dt_0 = c\int_0^t[h(t_0)\,\delta[x - ct + ct_0]\,dt_0$$
$$= c\int_0^t[h(t_0)\,\delta\left[c\left(t_0 - \left(t - \frac{x}{c}\right)\right)\right]dt_0$$
$$= \int_0^t[h(t_0)\,\delta\left(t_0 - \left(t - \frac{x}{c}\right)\right)dt_0 \quad \text{as} \quad \int_{-\infty}^{\infty}\delta(ax-b)\varphi(x)dx = |a|^{-1}\varphi(ba^{-1})$$

$$\Rightarrow u(x,t) = \begin{cases}0 & if\ x > ct\\ h\left(t - \frac{x}{c}\right) & if\ x < ct\end{cases} \quad (21)$$

since (i) $x > ct \Rightarrow t - \frac{x}{c} < 0$ and as $0 < t_0 < t$, so $\delta[x - ct + ct_0] = 0$

and (ii) $x < ct \Rightarrow 0 < t - \frac{x}{c} < t$ and as $0 < t_0 < t$ and $\int_{-\infty}^{\infty}\varphi(x)\delta(x-a)dx = \varphi(a)$

Thus, (21) provides the required solution to problem described in (16).

One more problem on semi-infinite domain, but now for $x < 0$, and with nonzero initial condition, is solved here, using same arguments.

Problem 2: Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$; $x < 0$ (22)

with $u(x,0) = \cos x$, $x < 0$; $\frac{\partial u}{\partial t}(x,0) = 0,\ x < 0$ and $u(0,t) = e^{-t}$, $t > 0$

As shown,

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} (G(x,t;x_0,t_0) \cdot Q(x,t))\, dx_0\, dt_0$$
$$+ \int_{-\infty}^{\infty} \left[\frac{\partial u}{\partial t_0}(x_0,0) G(x,t\,;x_0,0) - u(x_0,0) \left[\frac{\partial}{\partial t_0} G(x,t_0;x_0,t) \right]_{t_0=0} \right] dx_0$$
$$- c^2 \int_0^t [\, u(0,t_0) \left[\frac{\partial}{\partial x_0} G(x,t;x_0,t_0) \right]_{x_0=0} - G(x,t;\ 0,t_0) \left[\frac{\partial}{\partial x_0} u(x_0,t_0) \right]_{x_0=0}] \cdot \hat{n}\, dt_0$$

So, $u(x,t) = -\left\{ \int_{-\infty}^{0} [\cos x_0] \left[\frac{\partial}{\partial t_0} G(x,t_0;x_0,t) \right]_{t_0=0} dx_0 \right\} - c^2 \int_0^t e^{-t_0} \left[\frac{\partial}{\partial x_0} G(x,t;x_0,t_0) \right]_{x_0=0} dt_0$ (23)

As here $Q(x,t) = 0,\ u(x,0) = \cos x$ if $x < 0,\ \frac{\partial u}{\partial t}(x,0) = g(x) = 0,$ & $u(0,t) = e^{-t}$ and $(x,t;\ 0,t_0) = 0.$

Also, $\left[\frac{\partial}{\partial x_0} G(x,t;x_0,t_0) \right]_{x_0=0} = \frac{1}{c} \{ \delta[x - c(t-t_0)] - \delta[x + c(t-t_0)] \}$

Here in this case, since $x < 0 \Rightarrow x - ct < 0,$ ($c > 0$ and > 0), $\delta[x - c(t-t_0)]$ is always zero.

So, $\left[\frac{\partial}{\partial x_0} G(x,t;x_0,t_0) \right]_{x_0=0} = -\frac{1}{c}\, \delta[x + c(t-t_0)]$ (24)

Now $\frac{\partial}{\partial t_0} G(x,t_0;x_0,t) = \frac{1}{2c} \{ \frac{\partial}{\partial t_0} \{ -H[(x-x_0) - c(t-t_0)] + H[(x-x_0) + c(t-t_0)] +$
$H[(x+x_0) - c(t-t_0)] - H[(x+x_0) + c(t-t_0)] \}$
$= \frac{1}{2} \{ -\delta[(x-x_0) - c(t-t_0)] - \delta[(x-x_0) + c(t-t_0) + \delta[(x+x_0) -$
$c(t-t_0)] + \delta[(x+x_0) + c(t-t_0)] \}$

$$\Rightarrow \left[\frac{\partial}{\partial t_0} G(x,t_0;x_0,t) \right]_{t_0=0}$$
$$= \frac{1}{2} \{ \delta[(x+x_0) + ct] + \delta[(x+x_0) - ct] - \delta[(x-x_0) + ct] - \delta[(x-x_0) - ct] \}$$

Here, in this case $\delta[(x+x_0) - ct] = 0,$ for each x_0, as $-x - ct > 0$ and $-\infty < x_0 < 0$

So, $\left[\frac{\partial}{\partial t_0} G(x,t_0;x_0,t) \right]_{t_0=0} = \frac{1}{2} \{ \delta[(x+x_0) + ct] - \delta[(x-x_0) + ct] - \delta[(x-x_0) - ct] \}$ (25)

Substituting (25) and (24) in (23),

$$u(x,t) = -\left\{ \int_{-\infty}^{0} [\cos x_0] \left\{ \frac{1}{2} \{ \delta[(x+x_0) + ct] - \delta[(x-x_0) + ct] - \delta[(x-x_0) - ct] \} \right\} dx_0 \right\}$$
$$- c^2 \int_0^t e^{-t_0} \left[-\frac{1}{c} \{ \delta[x + c(t-t_0)] \} \right] dt_0$$

$$\Rightarrow u(x,t) = -\left\{ \int_{-\infty}^{0} [\cos x_0] \left\{ \frac{1}{2} \{ \delta[(x+x_0) + ct] - \delta[(x-x_0) + ct] - \delta[(x-x_0) - ct] \} \right\} dx_0 \right\}$$
$$+ c \int_0^t e^{-t_0}\, \delta[x + c(t-t_0)]\, dt_0$$

$$\Rightarrow u(x,t) = -\left\{ \int_{-\infty}^{0} [\cos x_0] \left\{ \frac{1}{2} \{ \delta[(x+x_0) + ct] - \delta[(x-x_0) + ct] - \delta[(x-x_0) - ct] \} \right\} dx_0 \right\}$$
$$+ \int_0^t e^{-t_0}\, \delta\left[\frac{x}{c} + (t-t_0) \right] dt_0 \quad (26)$$

If $x < -ct$, in the first integral on R.H.S of (26), $\delta[(x+x_0) + ct] = 0$ and in the second integral, $\delta\left[\frac{x}{c} + (t-t_0) \right] = 0.$

So for $x < -ct$, $u(x,t) = -\left\{ \int_{-\infty}^{0} [\cos x_0] \left\{ \frac{1}{2} \{ -\delta[(x-x_0) + ct] - \delta[(x-x_0) - ct] \} \right\} dx_0 \right\} + 0$

$$= \frac{1}{2} \{ \cos(x+ct) + \cos(x-ct) \} = \cos x \cos ct$$

If $x > -ct$, in the first integral on R.H.S. of (26), $\delta[(x-x_0) + ct] = 0.$

So for

$x > -ct, \quad u(x,t) = -\left\{\int_{-\infty}^{0} [\cos x_0] \left\{\frac{1}{2}\{\, \delta[(x+x_0)+ct] \; - \delta[(x-x_0)-ct]\}\right\} dx_0\right\} \; + \int_0^t e^{-t_0}\, \delta\left[\frac{x}{c} + (t-t_0)\right] dt_0$

$$= -\cos\big(-(x+ct)\big) + \cos(x-ct) + e^{-\left(t+\frac{x}{c}\right)} = \sin x \sin ct + e^{-\left(t+\frac{x}{c}\right)}$$

Thus

$$u(x,t) = \begin{cases} \cos x \cos ct & if \quad x < -ct \\ \sin x \sin ct + e^{-\left(t+\frac{x}{c}\right)} & if \quad 0 > x > -ct \end{cases} \qquad (27)$$

Thus (27) provides solution to problem-2.

III. CONCLUSION

One can realize benefits of Green's formula method to solve non-homogeneous wave equation as follows:

1. Result (10) provides solution in terms of Green's function. This formula covers all the three types of non-homogeneous terms. The first integral on R.H.S. shows effect of nonhomogeneous source term $Q(x,t)$ to the response, the second integral shows effect of initial conditions and the third integral shows effect of nonhomogeneous boundary condition. So any kind of non-homogeneous term can be treated by this single formula.
2. If boundary condition is non-homogeneous, there is no need to assume any function which satisfies related homogeneous boundary condition. In this method to deal with non-homogeneous boundary condition, only it is required to get the Green function which satisfies related homogeneous boundary condition. This point provides ease.

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